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Tensor product of Tate objects and higher adeles.

1 November 2011.

Joint with Braunling, Gröchenig, Wolfson.

I. Tate objects.

$\mathcal{C}$ an exact category.

Classes of admissible monos, epis.

Exs. (i) $\mathcal{C} = \mathrm{Mod}_R^{f.g.p}$, finitely generated projective R-modules.

(ii) $\mathcal{C} = \mathrm{Coh}(X)$, X some loc. noetherian scheme.

(i) $\mathrm{Ind}^a(\mathcal{C}) \simeq \{ X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots \}$

(ii) $\mathrm{Pro}^a(\mathcal{C}) \simeq \{ \cdots X_2 \twoheadrightarrow X_1 \twoheadrightarrow X_0 \}$

Def. $\mathrm{Tate}^{el}(\mathcal{C})$ is the smallest subcategory of $\mathrm{Ind}^a(\mathrm{Pro}^a(\mathcal{C}))$ containing $\mathrm{Ind}^a(\mathcal{C})$, $\mathrm{Pro}^a(\mathcal{C})$ and closed under extensions. This the category of elementary Tate objects.

Exs. (a) $R((t)) \simeq R[[t]] \oplus t^{-1}R[t^{-1}]$.

$R((t)) \simeq \operatorname*{colim}_{n \geq 1} \lim_{m \geq 1} \dfrac{t^{-n} R[[t]]}{t^m R[[t]]}$

Tate over R.

(b) X/k smooth curve over a field k

at any point $x \in X$,

$$\mathbb{A}_X = \prod_{x \in X}{}' \hat{K}_{X,x} \text{ --- Restricted product.}$$

completed local ring

Proposition. $\mathrm{Tate}^{el}(\mathcal{C}) \subseteq \mathrm{Ind}^a(\mathrm{Pro}^a(\mathcal{C}))$ is the collection of objects with a presentation $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$, with $X_i \in \mathrm{Pro}^a(\mathcal{C})$, $X_i / X_{i-1} \in \mathcal{C}$.

II. Tensor products.

Tate vector spaces over k $\subseteq$ topological vector spaces over k.

Setup $\mathcal{C}$ a symmetric monoidal exact category with a biexact tensor product.

Ex.

$$k((s)) \vec{\otimes} k((t)) \simeq \underset{n\geq 1}{\mathrm{colim}}\ \underset{m\geq 1}{\lim}\ \underset{k\geq 1}{\mathrm{colim}}\ \underset{l\geq 1}{\lim} \Big(\overbrace{k\langle t^{-n}, \dots, t^{m+1}\rangle}^{k\text{-vector space with basis } t^{-n},\dots,t^{m+1}} \underset{k}{\otimes} k\langle s^{-k}, \dots, s^{l+1}\rangle \Big).$$

$$\| \colon$$

$k((s))((t))$.

Rem. $k((s))((t)) \not\simeq k((t))((s))$ when viewed as objects of $\mathrm{Tate}^2(k)$.

Thm. There is an extension

$$\vec{\otimes} : \mathrm{Tate}^n(\mathcal{C}) \times \mathrm{Tate}^m(\mathcal{C}) \longrightarrow \mathrm{Tate}^{n+m}(\mathcal{C})$$

for $m,n \geq 0$ which recovers $\otimes$ on $\mathcal{C}$ for $n=m=0$. It is associative.

Rem. If we pass to $\mathrm{Tate}^{\infty}(\mathcal{C}) \simeq \bigcup \mathrm{Tate}^n(\mathcal{C})$, we get a monoidal structure.

What are other choices?

$V \in \mathrm{Tate}^2(\mathcal{C})$, $W \in \mathrm{Tate}^1(\mathcal{C})$

$\mathrm{colim}\ \lim\ \mathrm{colim}\ \lim (V_\cdot) \otimes \mathrm{colim}\ \lim (W_\cdot)$

Swapping leads to shuffle.

Continuation of Theorem. For any (n,m)-shuffle σ,

$$\vec{\otimes}_\sigma : \mathrm{Tate}^n \times \mathrm{Tate}^m \longrightarrow \mathrm{Tate}^{n+m}.$$

Rem. Taking the tensor product over all shuffles,

$$V \otimes_{\mathrm{all}} W \simeq \bigoplus_{\sigma} V \vec{\otimes}_{\sigma} W,$$

we get a symmetric monoidal category.

III. X/k a regular noetherian scheme. $\dim X = n$.

A full flag in X is $\{\eta_0 > \eta_1 > \cdots > \eta_n\} =: \Delta$, a sequence of scheme points, where $\eta_{i+1} < \eta_i$ iff $\eta_{i+1} \in \overline{\{\eta_i\}}$, and $\mathrm{codim}_X\left(\overline{\{\eta_i\}}\right) = i$.

Fix $\mathcal{F} \in \mathrm{Coh}(X)$.

$$A(\Delta, \mathcal{F}) := \lim_{n \geq 1} A\left(\Delta \setminus \{\eta_0\}, \frac{\mathcal{F} \otimes \mathcal{O}_{X,\eta_0}}{I^n_{\overline{\{\eta_0\}}}}\right),$$

and for $\mathcal{F} \in \mathrm{QCoh}(X)$,

$$A(\Delta, \mathcal{F}) \simeq \operatorname*{colim}_{\mathcal{G} \subseteq \mathcal{F} \text{ coherent}} A(\Delta, \mathcal{G}).$$

~~Exercise. $A(\Delta, \mathcal{F}) \in \mathrm{Tate}^n\left(\mathrm{Coh}(X_{\{\eta_n\}})\right)$~~

Exercise. $A(\Delta, \mathcal{F}) \in \mathrm{Tate}^n\left(\mathrm{Coh}(X)_{\overline{\{\eta_n\}}}\right)$.

$\underline{\text{Ex}}$. $Y = \mathrm{Spec}(k[t,s])$

$\Delta = \{ \eta_Y > \eta_C > \{0\} \}$

$[redacted]$ $C = \mathrm{Spec}\, k[s]$.

$$A(\Delta, \mathcal{O}_Y) \simeq \operatorname*{colim}_{n \geq 1} A(\Delta \setminus \eta_Y, t^{-n} s^{-n} k[t,s])$$

$$\simeq \operatorname*{colim}_{n \geq 1} \lim_{m} A\left(-, \frac{t^{-n}(k[t] \otimes k[s])}{t^{m}(k[t] \otimes k[s])}\right)$$

$$\simeq k(s) \vec{\otimes} k(t).$$

η, ξ points in X.

$${}_{\eta}\Omega_{\xi}(\mathcal{F}) = \operatorname*{colim}_{G \subset \mathcal{O}_{X,\xi}} \lim_{n \geq 1} \frac{G \otimes \mathcal{F}}{I^n_{\{\eta\}}}.$$

${}_{\{0\}}\Omega_{k(C)}(\mathcal{O}_Y) \simeq k(s)$

${}_{k(C)}\Omega_{k(Y)}(\mathcal{O}_Y) \simeq k(t)$.

<u>Thm.</u> X as above, Δ a full flag.

$$A(\Delta, \mathcal{O}_X) \simeq {}_{\eta_n}\Omega_{\eta_{n-1}}(\mathcal{O}_X) \vec{\otimes} \cdots \vec{\otimes} {}_{\eta_1}\Omega_{\eta_0}(\mathcal{O}_X).$$

<u>Rem.</u> If Y has $\dim Y = m$, $\mathcal{G}$ on $\mathrm{Coh}(Y)$, Π a full flag on Y,

$$A(\Delta, \mathcal{F}) \vec{\otimes} A(\Pi, \mathcal{G}) \simeq A\left(\eta_0 \times \xi_0 > \cdots > \eta_0 \times \xi_m > \eta_1 \times \cdots ;\ \mathcal{F} \boxtimes \mathcal{G}\right).$$

Shuffles enter via the different ways to produce a flag on $X \times Y$.